

Non-Stationary Additive Utility and Time Consistency

Nicolas Drouhin*

September 25, 2017

Abstract

Within a continuous time life cycle model of consumption and savings, I study the properties of the most general class of additive intertemporal utility functionals. They are not necessarily stationary, and do not necessarily multiplicatively separate a discount factor from “per-period utility”. I prove rigorously that time consistency holds if and only if the per-period felicity function is multiplicatively separable in t , the date of decision and in s , the date of consumption, or equivalently, if the Fisherian instantaneous subjective discount rate does not depend on t . The model allows to explain “anomalies in intertemporal choice” and various empirical regularities, even when the agents are time consistent. On the other hand, the model allows to characterize mathematically the “effective consumption profile” of naive, time-inconsistent agents.

Code JEL: D81 D91

Key words : intertemporal choice; optimal control, life cycle theory of consumption and saving; stationarity; time consistency; time invariance; exponential discounting; hyperbolic discounting; aging

*Ecole normale supérieure Paris-Saclay and CREST (UMR CNRS 9194)

1 Introduction

For a long time, conventional wisdom has considered Samuelson (1937)'s exponential discounting model as the only additive utility model compatible with a time-consistent consumption behavior. This paper studies the most general class of utility functions that are altogether additively time separable, not necessarily stationary and, nevertheless, imply full intertemporal rationality (time-consistent behavior). It shows that this class is broader than is generally assumed, comprising many simple cases that have never been used, and may be useful to settle some important economic problems such as, for example, the effect of aging on the intertemporal choice of consumption and savings. Thus, it hopes to clarify the notion of time consistency and, in particular, to give a better understanding of the distinction between stationarity and time consistency, which has recently been pointed out by Halevy (2015), and there concrete implications for applied economic modeling.

The notion of stationarity was introduced by Koopmans (1960, 1972), who studied choice between infinite-horizon consumption streams. If, for a given decision date, the preference order between any two consumption streams is not modified when those streams are anticipated or postponed by the same amount of time, then the preference relation fulfills stationarity. Fishburn and Rubinstein (1982) have given a definition of stationarity for choice between single dated outcomes, in which the infinity of the horizon is no longer required. Since Fishburn and Rubinstein (1982)'s settings fit well with common practice in experimental decision theory, their definition is now standard.

Stationarity is a form of weak separability that implies recursive utility (Blackorby et al., 1978). As shown in Koopmans (1960)'s seminal paper, additive separability and stationarity together imply the exponential discounting model. Thus, a first strategy for economists or psychologists dissatisfied with the exponential discounting model is to explore the more general recursive utility model of intertemporal choice that drop additivity and keep only stationarity (Epstein and Hynes, 1983, for example). But, as noted by Fishburn and Rubinstein (1982) "(...) we know of no persuasive argument for stationarity as a psychologically viable assumption." In particular, stationarity as such cannot be considered as an axiom of rationality. It is the reason why, in this article, we will take the opposite strategy to drop stationarity and keep additivity.

The notion of time consistency has been alluded by Ramsey (1928), Samuelson (1937) and Allais (1947) and was formally introduced by Strotz (1956) and clarified by Blackorby et al. (1973). To define this notion properly, it is important to distinguish (1) the *calendar date of decision (planning)* (denoted t) from (2) the *calendar date of the future act of consumption* (denoted s). The agent will be *time consistent* if, in the absence of any new information, the choice of consumption for any future calendar date is independent from the calendar date of decision. Clearly, time consistency is a criteria of rationality and as such is of great importance. Strotz's main claim was that only models that discount per-period utility by an exponential function of the algebraic time distance were time consistent.

Since stationarity and time consistency both give foundations to the exponential discounting model, the two notions have been progressively mixed in a sort of *conventional wisdom* that sometimes abusively sums up results which do not have the same domain of validity and have been established using different methodologies. As pointed out for example by Harrison et al. (2005) and Halevy (2015), it is now time to clearly disentangle the two notions. In particular, Halevy (2015) introduces a third notion, *time invariance*.

An intertemporal choice fulfils *time invariance* if the order of preference between two dated consumptions is not modified when the calendar date of decision and the date of both consumptions are postponed by the same amount of time. In other words, *time invariance* means that intertemporal preferences do not change with the calendar date of decision. Halevy (2015) stated the notion in the Fishburn and Rubinstein (1982) framework, but it can be easily generalized to consumption streams. The main theoretical result of Halevy (2015) is that, if intertemporal preferences fulfil any two of the following three properties : *time consistency*, *stationarity*, and *time invariance*, then they will also necessarily fulfil the third one. An immediate corollary of this proposition follows: if intertemporal preferences fulfil any one of the three properties and not any one of the two remaining, then they will also necessarily not fulfil the third one. This means that intertemporal preferences may be time consistent, but not stationary, if they are not time invariant. My aim is to study such intertemporal preferences, characterize the properties of utility functions that represent them, and draw direct implications for applied economic theory in the domain of the intertemporal choice of consumption and savings.

In this paper, I will not use the upstream, axiomatic methodology of Fishburn and Rubinstein (1982) and Halevy (2015), but instead, the downstream “choice-based” methodology which compares the solutions of dynamic programs with different decision dates. It is the methodology used originally by Strotz (1956), and now standard in behavioral macroeconomics, since the pioneering work of Laibson (1994, 1997). It is “choice based” because it not only uses a utility function that represents the preference relation, but also the budgetary constraints that the decision maker faces. As we will see, to study intertemporal choices, because of the dynamic nature of the constraint, past consumption choices will affect future choices even when preferences fulfil a coordinate independence axiom (see Wakker, 1989, for a definition).

However, the purpose of Laibson, relying on the special case of quasi-hyperbolic discounting, was to illustrate the power of time inconsistent preferences to solve many empirical puzzles (Laibson, 1998). Our goal, is different. We want to characterize the the most general properties of intertemporal utility functionals implying time consistency within the additive framework. We want to clarify, once for all, the boundaries between fully rational intertemporal choice model and time inconsistent ones. Of course the question is of first importance, because beyond the boundaries of full rationality, the ways to think at economic policy is deeply modified introducing the possibility of *libertarian paternalism* (Thaler and Sunstein, 2003) (see Saint-Paul, 2011, for an extensive discussion).

Within a typical modern macro model, with random shocks at each period, a rational agent has to change her plan at each period to take into account the new information provided by the realisation of current shocks. That is the essence of Hall (1978) random walk result for life-cycle consumption. On the opposite, in absence of any uncertainty, only a non rational, time inconsistent, agent will change her plan over-time. A model, that introduce time inconsistency in a buffer stock model of savings, characterized by random income and borrowing constraint (see Harris and Laibson, 2001, for a rigorous resolution) may certainly be more realistic and interesting, but it prevent from using the notion of changing plans to dichotomize between time consistent and time inconsistent model. In other words, time inconsistency implies a kind of “endogenous change of choice” over time, so it is important to sterilize any “exogenous sources” that may explain changing choices. That is why it is important to work with a model with no uncertainty.

Moreover, to study time consistency, it is crucial to work with many periods. Discrete time models, with vectors of dated consumption, can rapidly become cumbersome. Con-

tinuous time allows to deal with intertemporal consumption profiles, which are just one variable functions of time, a much more tractable mathematical object. In this paper, I will develop a continuous time life-cycle model of consumption and savings, without uncertainty. I will use optimal control to solve the model, instead of the recursive approach of dynamic programming, that is better suited to deal with random shocks. Optimal control allows to obtain directly closed-form solutions, that are easiest to interpret.

I define the most general possible set of intertemporally additive utility functionals, allowing the per-period felicity function to vary according to the consumption date and the decision date. Within this set, I search for the special functional forms which are compatible with time consistency. The main result of the paper is that any function of the form $V(c, t) = \int_t^T w(c(s), s)ds$ implies time-consistent choices. Obviously, the exponential discounting model is a special case, the only one that also fulfills stationarity. All the other cases are additive, non-stationary and, nevertheless, time consistent. Renewing with the Fisherian tradition, I define the discount factor as a marginal rate of substitution, and derive a notion of instantaneous subjective discount rate that is altogether simple and insightful, and generalizes all the known definition of the discount rate. I then have all the material to discuss additional restrictions to the intertemporal utility functional which are required to explain some empirical regularities coming from experimental economics, or from the life-cycle model of consumption and savings. In particular, I reinterpret the so called “anomalies in intertemporal choice” as behavioral requirements for the utility function, implying some restrictions on the third derivatives of the per-period felicity function. Finally, I fully characterize the general properties of the observable consumption behavior of a naive non-time consistent agent, and illustrate them in a calibrated version of Barro (1999) continuous time model of a quasi-hyperbolic discounting consumer.

The rest of the paper is structured as follows. Section 2 describes the model, discusses the possible form of the intertemporal utility function, defines the generalized subjective discount rate and characterizes the general form of the solution of the maximization program. Section 3 gives the formal definition of time consistency and derives the core theorem of the paper. Section 4 reinterprets empirical findings on intertemporal discounting, within the time-consistent framework as behavioral requirements for the utility function. Part 5 characterizes the general mathematical properties of the effective consumption profile for time non-consistent agents. Finally, part 6 concludes.

2 The Model

Time is continuous. Let us denote the “calendar date of decision” by t . A consumer is endowed with an initial capital, $K(t)$, and a planning horizon $T > t$. At every moment, this capital brings interest at a constant rate, r , and can be used to finance consumption.

At date t , the consumer has to decide the level of consumption $c(s)$ for any further date $s \in [t, T]$. Thus s refers to the “calendar date of consumption”. If the birthdate of the agent is normalized to zero, $c(s)$ can also be interpreted as the consumption at age s^1 . \mathcal{C}_t will be considered the set of all functions $c : [t, T] \rightarrow \mathbb{R}^+$ continuous and derivable.

I assume that the consumer’s intertemporal preferences at date t are represented by

¹Although both t and s can be interpreted as age, in the rest of the paper the word age will only be used to refer to the time of consumption.

an additive utility functional,

$$V(c, t) = \int_t^T v(c(s), s, t) ds \quad (1)$$

with v , the “per-period felicity function”, three times continuously differentiable in $c(s)$ and s , once continuously differentiable in t , and strictly increasing and strictly concave in $c(s)$. I also assume that $v_1(c, s, t) \rightarrow -\infty$ when $c \rightarrow 0$. The set of all intertemporal utility functionals of that kind defined on $\mathcal{C}_t \times [t_0, T]$ is denoted \mathcal{V}_t , with t_0 the minimal possible value for t .

The specification of the utility functional (1) is as general as possible within the additive framework. It means that it includes, as special cases, all the possible continuous additive utility models of intertemporal choice, and especially the standard exponential discounting model and many² of the special hyperbolic discounting models proposed in the literature in the last three decades. It also covers many cases that have not been investigated yet.

Moreover, being altogether more general and simpler, this formulation of the intertemporal utility functional will allow to point out some important results and interpretations, which were hidden when using more specific functional form. As such, this formulation is the main conceptual innovation of this article, all the following results stemming directly from it.

It is important to notice that this general formulation encompasses two important notions, *time discounting* and the possibility of *changing tastes*.

From the point of view of time discounting, since at least Samuelson (1937), there has been a tradition in economics of using per-period felicity functions that multiplicatively separate a discount factor (with dates as variable) from per-period utility (with consumption levels as variables). This is obvious in Samuelson (1937)’s exponential discounting model, in which the subjective discount factor is exponential (i.e. the subjective discount rate is constant). But one can find some more general models, for example Yaari (1964), Harvey (1986), or Harvey and Østerdal (2012), in which the discount function can be any continuous function of s . In those models, the discount rate is just the log derivative of the discount factor. In the general model of this article, characterized by equation (1), this separability is not postulated. However, going back to Fisher (1930) both discount factor and rate can easily be defined.

For Irving Fisher, the discount factor is just the marginal rate of substitution between two same levels of consumption at different dates. If the constant fixed level of consumption is denoted by c and the time distance between the two dates by τ , we have:

$$e^{-\Theta(c, s, \tau, t)\tau} = \frac{v_1(c, s + \tau, t)}{v_1(c, s, t)} \quad (2)$$

Taking the limit of this expression when τ goes to 0, we get the definition of the instantaneous subjective discount rate:

$$\Theta(c, s, t) = -\frac{1}{v_1(c, s, t)} \lim_{\tau \rightarrow 0} \frac{v_1(c, s + \tau, t) - v_1(c, s, t)}{\tau}$$

²Even if formulated within a discrete time framework, the quasi-hyperbolic model proposed by Laibson (1997) (borrowed from Phelps and Pollak (1968)) appears to have a kind of discontinuity between the utilities of present and future consumptions. However, this pattern can always be approximated by a continuous time model, as shown by Barro (1999)(see Section 5 for further details). Harris and Laibson (2013) and Pan et al. (2015) propose alternative transpositions of quasi-hyperbolic discounting in continuous time.

Definition 1. The *instantaneous subjective discount rate* is defined as:

$$\Theta(c, s, t) \stackrel{def}{=} -\frac{v_{12}(c, s, t)}{v_1(c, s, t)} \quad (3)$$

The instantaneous subjective discount rate is just the rate at which the marginal felicity of a given future consumption changes with age. It is easy to verify that when $v(c, s) = \text{Exp}(-\theta s)u(c)$ (the exponential discounting model), the instantaneous subjective discount rate is θ , proving that my definition³ is a generalization of the usual one. It is also noteworthy that, until now, no assumption has been made about the sign of the second-order cross derivative of v . As long as the agent's horizon is finite, such an assumption is not technically required to have an optimum for the agent's program. Thus my model can explain *preference for present consumption*, when $w_{12} < 0$, as well as *preference for future consumption*, $w_{12} > 0$. However, it is important to notice that preference for present consumption is not formally characterized by a per-period felicity function that is decreasing with s , but by a marginal felicity of consumption that is decreasing with s . With the general utility functional characterized by equation (1), the subjective instantaneous discount rate may vary according to the level of consumption, the age, and the calendar date of decision. It will be the same for the notion of resistance toward intertemporal substitution. This point will be of first importance when discussing time consistency in the next section.

Definition 2. The *rate of absolute resistance to intertemporal substitution* is defined as:

$$\rho(c(s), s, t) \stackrel{def}{=} -\frac{v_{11}(c(s), s, t)}{v_1(c(s), s, t)} \quad (4)$$

This second definition is standard. What is noteworthy is the symmetry between the definitions of *the instantaneous subjective discount rate* and *the rate of absolute resistance to intertemporal substitution*. The two rates measure the curvature of the relation between the marginal felicity of consumption and each one of its determinant taken separately. I will show that, similarly to the standard exponential discounting model, these two rates, jointly with the rate of interest, will determine the slope of the intertemporal consumption profile.

The fact that the rate of absolute resistance to intertemporal substitution can vary with the level of consumption has been discussed extensively in the literature⁴. But the possibility for this rate, for a given level of consumption, to vary according to age and decision date has generally been discarded as implying irrational behavior. Before moving on to challenge this last point, I have to discuss the possibility for consumers to change tastes, induced by the formulation of the utility functional (1).

Taking s as a variable of the "per-period felicity function" allows me to include the possibility of tastes evolving with age in the analysis. Obviously, this assumption may imply a departure from *time invariance* and *stationarity*, because in this case postponing or anticipating two different future consumption streams can clearly alter the order of preference of these streams. In this model, the possibility that tastes for future consumption will evolve with age s is clearly anticipated by the decision maker at decision date t .

³The same kind of definition is used by Gollier and Zeckhauser (2005), but they do not make the link with the Fisherian definition.

⁴In the case of risk, the fact that the utility function is DARA, is generally considered as realistic (see Gollier (2001) for an extensive discussion.

Taking t as a parameter of the “per-period felicity function” allows me to include the possibility of changing tastes at different calendar dates of decision in the analysis. This other possibility of changing tastes is more “drastic” than the former. Clearly, this formulation allows the decision maker, if given the possibility, to change her mind in the future and to decide for a different stream of consumption. In this paper, I will consider a naive decision maker. Either because, following Strotz (1956) she does not anticipate in any way that she will be allowed to plan again at any subsequent date, or because she does not anticipate the possibility that her preferences for future consumption can change according to t . I will discuss this point further in the next section.

The choice of the consumer at date t is the function $c_t \in \mathcal{C}_t$ solution of the program:

$$\mathcal{P}_t \quad \begin{cases} \max_c \int_t^T v(c(s), s, t) ds \\ s.t. \forall s \in [t, T], \quad \dot{K}(s) = rK(s) - c(s) \\ K(t) \text{ given and } K(T) \geq 0 \end{cases}$$

Thus, $c_t(s)$ is the optimal consumption at age s , planned at date t (the value of the control variable at age s), and $K_t(s) = K(t) + \int_t^s (rK(\tau) - c_t(\tau)) d\tau$, the remaining capital at age s for an agent planning her consumption at date t (the value of the state variable at age s).

Proposition 1. *The optimal consumption profile planned at date t (i.e. the solution of the program \mathcal{P}_t) is fully described by:*

$$\dot{c}_t(s) = \frac{r - \Theta(c_t(s), s, t)}{\rho(c_t(s), s, t)} \quad (5)$$

$$\int_t^T \exp[-r(s-t)] c_t(s) ds = K(t) \quad (6)$$

Proof: The Hamiltonian of the program \mathcal{P}_t is $\mathcal{H}(c, K, s, t) = v(c, s, t) + \lambda(rK - c)$. The maximum principle gives us the necessary condition for c_t and K_t to be the solution of \mathcal{P}_t . $\forall s \in [t, T]$:

$$\lambda(s) = v_1(c_t(s), s, t) \quad (7)$$

$$\dot{\lambda}(s) = -r\lambda(s) \quad (8)$$

$$\dot{K}_t(s) = rK_t(s) - c(s) \quad (9)$$

$$\lambda(T)K_t(T) = 0 \quad (10)$$

Differentiating (7) according to s and substituting in (8) gives (5). The strict positive monotonicity of v according to c and condition (7) imply that $\lambda(s) > 0$ and then, taking into account the transversality condition (10) proves that $K_t(T) = 0$. Multiplying (9) by $\exp[-r(s-t)]$ and integrating by parts on the interval $[t, T]$ gives (6). \square

Proposition 1 generalizes the standard results of the theory of consumption and savings (Yaari, 1964), making it very insightful. Equation (5) characterizes the *slope* of the optimal consumption profile and equation (6) its *level*.

The *slope* depends mainly on the psychological characteristics of the agent embodied in the intertemporal utility function. These characteristics are properly summarized by the two concepts of instantaneous subjective discount rate and rate of absolute resistance to intertemporal substitution. The key issue is the difference between the economical

discount rate, r , and the subjective rate of discount. If $r > \Theta$, there is an incentive for the agent to consume more in the future. The slope will be positive. On the contrary, if $r < \Theta$, there is an incentive to consume more in the present and thus the slope will be negative. If $r = \Theta$, the intertemporal consumption profile will be flat. The intensity of the agent's response to a difference between the economic and the subjective discount rates will be determined by the rate of resistance to intertemporal substitution. In my model, the fact that both rates can vary with the consumption level and with age implies that the dynamic of consumption will be richer.

The *level* of the optimal consumption profile only depends on life-cycle wealth. In this article, it is simply the capital at decision date t .

Equation (5) describes the consumption profile planned at date t as a differential equation. At some point in the article, it will be more convenient to specify the consumption profile in an integral form. If the instantaneous rate of growth at age s of the consumption profile planned at date t is denoted $G(t, s)$, we have:

$$G(t, s) \stackrel{\text{def}}{=} \frac{r - \Theta(c_t(s), s, t)}{\rho(c_t(s), s, t)c_t(s)} \quad (11)$$

Corollary 1 (The optimal consumption profile integral form).

$$c_t(s) = c_t(t) \exp \left[\int_t^s G(t, \tau) d\tau \right] \quad (12)$$

Proof: From equation (5) taken for all $\tau \in [t, T]$, we get:

$$\frac{\dot{c}_t(\tau)}{c_t(\tau)} = \frac{r - \Theta(c_t(\tau), \tau, t)}{\rho(c_t(\tau), \tau, t)c_t(\tau)} = G(t, \tau)$$

Integrating this relation between t and s , we get:

$$\ln(c_t(s)) - \ln(c_t(t)) = \int_t^s G(t, \tau) d\tau$$

Taking the exponential, equation (12) follows directly. \square

It is important to note that equation (12) is not an explicit formulation of c , but an implicit one, because both sides of the equation depend on $c_t(s)$.

Having all the necessary material now, I may turn to the main concern of this paper, time consistency.

3 Time Consistency for a Naive Decision Maker

Let us now consider a naive decision maker who is allowed to plan her consumption again at date $t' > t$. Her optimal choice $c_{t'} \in \mathcal{C}_{t'}$ is the solution of the program:

$$\mathcal{P}_{t'} \quad \begin{cases} \max_c \int_{t'}^T v(c(s), s, t') ds \\ s.t. \forall s \in [t', T], \quad \dot{K}(s) = rK(s) - c(s) \\ K(t') = K_t(t') \text{ and } K(T) \geq 0 \end{cases}$$

Time consistency can now be properly defined.

Definition 3. The agent is **time consistent** if and only if, for all possible interest rate, r , and initial value of capital, $K(t)$:

$$\forall t' \in [t, T], \forall s \in [t', T], c_t(s) = c_{t'}(s) \equiv c(s)$$

This definition is exactly the same nature as the one given by Strotz (1956), so is the methodology. Like Strotz, within a set of utility functionals, I will seek the ones that are compatible with time consistency. The only difference is that we are searching in the set \mathcal{V}_t , the one described by Equation 1, apparently a much broader set than the one used by Strotz.

Within this set, what are the specificities of the utility functions compatible with time-consistent behavior?

Proposition 2. The three following statements are equivalent:

1. The agent is time consistent.
2. The per-period felicity function v is of the form:

$$v(c(s), s, t) = \alpha(t)w(c(s), s) + \gamma(s, t) \quad (13)$$

with w any real valued function defined on $\mathbb{R}^+ \times [t, T]$, three times continuously differentiable in c , s , and strictly concave in c ; and α and γ any continuous real valued function defined on $[t, T] \times [t_0, T]$.

3. The instantaneous subjective discount rate and the rate of absolute resistance to intertemporal substitution are both independent from t , the calendar date of decision.

$$\Theta(c(t), s, t) = -\frac{w_{12}(c(s), s)}{w_1(c(s), s)} \text{ and } \rho(c(t), s, t) = -\frac{w_{11}(c(s), s)}{w_1(c(s), s)}$$

Proof: The solutions of the programs \mathcal{P}_t and $\mathcal{P}_{t'}$ over the interval $[t', T]$ must be compared for any possible values of r and $K(t)$. Because of the strict concavity of v according to c , these solutions are unique. They both satisfy their respective differential budgetary constraint. Discounting at the market rate r and integrating these constraints over the interval $[t', T]$, taking into account the fact that $K_t(T) = K_{t'}(T) = 0$ (cf. the proof of Proposition 1), we obtain:

$$K(t') = \int_{t'}^T \exp[-r(s - t')]c_t(s)ds = \int_{t'}^T \exp[-r(s - t')]c_{t'}(s)ds \quad (14)$$

Moreover, Equation (5) of Proposition (1) implies:

$$\dot{c}_t(s) = \frac{r + \frac{v_{12}(c_t(s), s, t)}{v_1(c_t(s), s, t)}}{-\frac{v_{11}(c_t(s), s, t)}{v_1(c_t(s), s, t)}} \text{ and } \dot{c}_{t'}(s) = \frac{r + \frac{v_{12}(c_{t'}(s), s, t')}{v_1(c_{t'}(s), s, t')}}{-\frac{v_{11}(c_{t'}(s), s, t')}{v_1(c_{t'}(s), s, t')}}$$

If $\forall s, c_t(s) = c_{t'}(s)$, then one must have $\forall s, \dot{c}_t(s) = \dot{c}_{t'}(s)$. Put differently, it means for all r, c and s , the derivative of the planned consumption stream does not depend on t . In this case, I denote this derivative as an arbitrary function Γ of the consumption, the consumption date, and the rate of interest. Γ is of the form: $\Gamma(c, s, r) = \tilde{A}(c, s, t) + \tilde{B}(c, s, t)r$, with \tilde{A} a real valued function and \tilde{B} a strictly positive real valued function.

Γ does not depend on t . So we have $\tilde{A}_3(c, s, t) - r\tilde{B}_3(c, s, t) = 0$. In particular this is true when $r = 0$ and $r = 1$, so we can deduce that $\tilde{A}_3(c, s, t) = 0$ and $\tilde{B}_3(c, s, t) = 0$. Thus Γ can be rewritten $\Gamma(c, s, r) = A(c, s) + B(c, s)r$, with A a real valued function and B a strictly positive real valued function.

Thus the time consistent utility functional should solve the following Partial Derivative Equation for every possible values of r :

$$rv_1(c, s, t) + v_{12}(c, s, t) = -(A(c, s) + rB(c, s))v_{11}(c, s, t)$$

In particular, this is true for $r = 0$ and $r = 1$, thus:

$$\begin{cases} v_1(c, s, t) &= -B(c, s)v_{11}(c, s, t) \\ v_{12}(c, s, t) &= -A(c, s)v_{11}(c, s, t) \end{cases}$$

And because B is always strictly positive:

$$\begin{cases} \frac{\partial}{\partial c} \text{Log } v_1(c, s, t) &= -\frac{1}{B}(c, s) \\ \frac{\partial}{\partial s} \text{Log } v_1(c, s, t) &= \frac{A}{B}(c, s) \end{cases}$$

and then:

$$\begin{cases} \text{Log } v_1(c, s, t) &= \int_0^c -\frac{1}{B}(k, s)dk + L(s, t) \\ &= \int_0^s \frac{A}{B}(c, \tau)d\tau + M(c, t) \end{cases}$$

With L and M two arbitrary functions. It gives:

$$\int_0^c -\frac{1}{B}(k, s)dk - \int_0^s \frac{A}{B}(c, \tau)d\tau + L(s, t) - M(c, t) = 0$$

The two integrals does not depend on t , so we have:

$$\frac{d}{dt}(L - M) = 0$$

It implies that L is of the form:

$$L(s, t) = M(c, t) + N(c, s)$$

with N an arbitrary function. Thus we have:

$$\frac{d}{dc}L = 0$$

And then:

$$M_1(c, t) = -N_1(c, s)$$

Which implies:

$$M_{12}(c, t) = N_{12}(c, s) = 0$$

That proves that M and N , and thus L , are necessarily additively separable:

$$\begin{cases} M(c, t) &= \gamma^1(c) + \gamma^3(t) \\ N(c, s) &= -\gamma^1(c) + \gamma^2(s) \\ L(s, t) &= \gamma^2(s) + \gamma^3(t) \end{cases}$$

With γ^i arbitrary functions.

Finally we can write:

$$v_1(c, s, t) = \exp \left[\int_0^c -\frac{1}{B}(k, s)dk + \gamma^2(s) + \gamma^3(t) \right]$$

Denoting, $w_1(c, s) = \exp \left[\int_0^c -\frac{1}{B}(k, s)dk + \gamma^2(s) \right]$ and $\alpha(t) = \exp [\gamma^3(t)]$, we get:

$$v_1(c, s, t) = \alpha(t)w_1(c, s)$$

Integrating, we get (13) which is thus necessary for c_t and $c_{t'}$ to have the same derivative over the interval $[t', T]$ for all possible values of r . Because of (14), they also have the same discounted integral on the interval $[t', T]$, they are thus equal on $[t', T]$. This proves Proposition 2.

□

Proposition 2 is the core proposition of this paper. The equivalence between 1. and 3. gives a practical way to test for time consistency. The subjective discount rate and the rate of resistance toward intertemporal substitution have to be independent from the *calendar date of decision*. But time consistency has nothing to do with the fact that these rates are invariant with the consumption level or with age. This has two important consequences. First, one can no longer consider that variations of the subjective discount rate according to the consumption level or according to the consumption date prove irrational behavior or anomalies in intertemporal choice. This point will be discussed further in the next section. Second, the only way to prove time inconsistency empirically on the basis of experimental measures of the subjective discount rate is to perform longitudinal experiments. The discount rate and the rate of resistance to intertemporal substitution (or rate of risk aversion) must be evaluated at different decision dates t . As noted by Sayman and Onculer (2009), until very recently, there have been very few longitudinal measurements of the discount rate.

Among the set of time inconsistent preferences, the literature has focused on preference that are “present biased”. According to O’Donoghue and Rabin (1999), “when considering trade-offs between two future moments, *present-biased* preferences give stronger relative weight to the earlier moment as it gets closer”. Taking into account the equivalence between 1. and 3. in proposition 2, a natural transposition of this statement in our general framework is the following formal definition:

Definition 4. *The agent is said to be **present bias** if:*

$$\forall t[t_0, T], \forall s \in [t, T], \forall c, \Theta_3(c(s), s, t) \geq 0, \text{ with } \Theta_3(c(t), t, t) > 0 \quad (15)$$

The equivalence between 1. and 2. is certainly the most substantial part of proposition 2. It implies that for an additive intertemporal utility functional to be time consistent, it requires only that the per-period felicity function be multiplicatively separable in t , the calendar date of decision, and in s , the date of consumption. It implies that all preferences that can be represented by a utility function of the kind $V(c, t) = \int_t^T w(c(s), s)ds$ are time consistent. If we substitute the per-period felicity function described in Equation (13) in the intertemporal utility functional, we obtain $V(c, t) = \alpha(t) \int_t^T w(c(s), s)ds + (T - t)\gamma(s, t)$. It is just a linear transformation of the former and thus represents exactly the same preferences.

“A infratemporal utility functional can be non stationary, additive and compatible with time consistency.” “Exponential discounting is not the only additive infratemporal

utility functional compatible with time consistency.” “Some hyperbolic discounting functions are compatible with time consistency.” All these statements stem from Proposition 2. They are paradoxical because they confront some largely shared opinions⁵. In fact, these false opinions stem from a kind of fallacy of composition, that will be made clear by the following corollary. My point is that the subset of time-consistent utility functionals critically depends on the nature of the set of functionals you’re searching in. Since I started from \mathcal{V}_t , a broader set of utility functionals than the one implicitly considered in the literature, it is no surprise that I have found a broader subset of time-consistent utility functionals.

As a corollary of Proposition (2), a subset X of \mathcal{V}_t can be used. I will denote $TC(X)$ the subset of all functionals belonging to X which also implies time consistency (*i.e.* $TC(X) = \{V(c, t) \in X | \forall s \in [t, T], v(c(s), s, t) = \alpha(t)w(c(s), s) + \gamma(s, t)\}$).

Corollary 2 (Special cases). *The following statements are true:*

- a. For $A = \{V \in \mathcal{V}_t | V(c, t) = \int_t^T f(s-t)u(c(s))ds\}$,
 $TC(A) = \{V \in \mathcal{V}_t | \int_t^T \exp[-\theta(s-t)]u(c(s))ds\}$
- b. For $B = \{V \in \mathcal{V}_t | V(c, t) = \int_t^T f(s, t)u(c(s))ds\}$,
 $TC(B) = \{V \in \mathcal{V}_t | \int_t^T \alpha(t)\beta(s)u(c(s))ds\}$
- c. For $C = \{V \in \mathcal{V}_t | V(c, t) = \int_t^T f(s-t)u(c(s), s)ds\}$,
 $TC(C) = \{V \in \mathcal{V}_t | \int_t^T \exp[-\theta(s-t)]u(c(s), s)ds\}$

We have $A \subset B \subset \mathcal{V}_t$ and $A \subset C \subset \mathcal{V}_t$ and thus, for obvious reasons, $TC(A) \subset TC(B) \subset TC(\mathcal{V}_t)$ and $TC(A) \subset TC(C) \subset TC(\mathcal{V}_t)$, demonstrating the increasing generality of the corollary.

$TC(A)$ is the set of all the exponentially discounted intertemporal utility functionals introduced by Samuelson (1937), the only one that is conventionally assumed to be compatible with time consistency. Assuming that the “per-period” felicity function multiplicatively separates a discount function that depends on the algebraic time distance between t and s from a *per-period* utility function that is the same at all ages, then only exponential discounting is time consistent. Considering Proposition 2, the result is obvious because only exponential functions “transform sums into products”. The important point to notice is that this case is very special, stemming from a rather arbitrary starting point. In light of Halevy (2015), it is easy to show that utility functionals belonging to A fulfil time invariance. Thus, it is no surprise that *time consistent* utility functionals of this kind will also fulfil *stationarity* and thus, being additive, correspond to Samuelson (1937)’s exponential discounting model.

If the separability between discounting and per-period utility function is preserved, but with a more general discount function, then we obtain Corollary 2-b. This result was demonstrated first by Burness (1976) (Theorem 1), but unfortunately has received very little attention until now. This case is interesting because it is intuitively close to the standard one, but it opens a broad range of time-consistent utility functionals which contradict conventional wisdom very clearly. The most striking example is the case

⁵To be right, we do not claim, that this opinion is universally shared. And a careful reading of the literature, allows to find some literary digression that convey some the ideas of Proposition 2. for example Saint-Paul (2011).

of hyperbolic discount functions satisfying time consistency given by Drouhin (2009). Indeed, $\int_t^T \frac{t}{s} u(c(s)) ds$ belongs to $TC(B)$. The discount factor is t/s . It decreases from 1, at date t , to 0 when s tends to infinity. The discount rate is $1/s$. It is decreasing with age and time distance, *i.e.* the agent shows decreasing impatience (Prelec, 2004). Nevertheless, an agent with such preferences is totally time consistent. It can be shown that utility functionals belonging to B are not necessarily *time invariant*, which explains why $TC(B)$ includes non-stationary intertemporal utility functionals.

The first demonstration of Corollary 2-a is generally attributed to Strotz (1956). But a careful reading shows that Strotz did not assume *time invariance* in the sense of Halevy (2015). Corollary 2-c corresponds exactly to the seminal results demonstrated by Strotz (1956). C is the set of utility functionals that multiplicatively separate a discount factor, but with a per-period felicity function that can vary with age. If Proposition 2 is applied directly, we can immediately deduce that the discount function has to be multiplicatively separable between s and t , implying the exponential form. But it is important to notice that per-period utility can always be rewritten with $u(c(s), s) = w(c(s), s) \exp[\theta s]$, with $W(c, t) = \int_t^T \exp[-\theta(s - t)] w(c(s), s) ds$ also belonging to C . Thus Strotz's time-consistent utility functional characterized by corollary 2-c can be rewritten

$$V(c, t) = \exp[\theta t] \int_t^T w(c(s), s) ds$$

proving that, in this case, the exponential formulation is a mathematical artefact. In other words, $TC(C) = TC(\mathcal{V}_t)$.

Another way to shed light on this point is to note that the parameter θ of the Strotzian formulation *is not* the Fisherian instantaneous subjective discount rate. Applying equation (3), we get:

$$\Theta(c(s), s, t) = \theta - \frac{u_{12}(c(s), s)}{u_1(c(s), s)}$$

The formulation of Strotz (1956)'s main result has proved misleading, in the sense that most subsequent works on the question have focused on the exponential formulation (the mathematical artefact) and forgotten the possible variability of the "per-period" utility function with age.

4 Empirical Considerations on the Shape of the Intertemporal Utility Functional

Proposition 2 is very general and opens a broad range of possibilities to model intertemporal choice. However, one can even find that the result is "too general". In particular, more precise specifications of the intertemporal utility functional may be wanted to do comparative statics or dynamics of the model. In this section, I will discuss the shape of the utility function, considering stylized facts from the empirical literature. Two literatures can be referred to: the experimental literature in decision theory, on the one hand, and the econometrical estimation of life-cycle consumption and savings, on the other hand.

Let us start with the experimental literature. Since the work of Ainslie (1975), much evidence has been collected in the lab against the descriptive power of Samuelson (1937)'s exponential discounting model. More precisely, following Thaler (1981), Loewenstein

and Prelec (1992) have proposed a typology to depart from the exponential discounting model, presented as “anomalies in intertemporal choice”. In the last two decades of the twentieth century, many experimental works (see Frederick et al. (2002) for a survey) have emphasized the possibility that the subjective discount rate is decreasing with time distance (*the common difference effect*) or with the level of consumption (*the absolute magnitude effect*). These empirical findings have been one of the starting points for what is sometimes called *the behavioral economics revolution*. Even if more recent experimental studies with appropriate protocols and incentive schemes do not find evidence of these anomalies (see Andreoni and Sprenger, 2012; Andersen et al., 2013, 2014, for exemple), it can be interesting to consider their logical implications within the the general framework of this article.

Usually, these anomalies were interpreted as stemming from time inconsistency. However, almost all of the experimental studies at the time were made with experimental protocols in which the decision date (t , with my notation) is invariant, and only the date of the future prospects (s) varies. As shown, the fact that the discount rate can vary with the date of the prospect proves absolutely nothing with regard to time consistency. Thus, in this section, I will interpret these anomalies as resulting from the rational choice of a time-consistent agent with non-stationary utility. Under this interpretation, I will characterize the restrictions necessary on the shape of the intertemporal utility functional to demonstrate these anomalies⁶.

The results are as follows:

Proposition 3 (Behavioral restrictions for the utility function of a time-consistent agent). *A time-consistent agent with preference for present consumption ($w_{12} < 0$):*

a. demonstrates the common difference effect if and only if

$$-\frac{w_{122}(c(s), s)}{w_{12}(c(s), s)} > \Theta(c(s), s)$$

b. demonstrates the absolute magnitude effect if and only if

$$-\frac{w_{121}(c(s), s)}{w_{12}(c(s), s)} > \rho(c(s), s)$$

c. demonstrates absolute resistance to intertemporal substitution decreasing with age if and only if

$$-\frac{w_{112}(c(s), s)}{w_{11}(c(s), s)} > \Theta(c(s), s)$$

d. The absolute magnitude effect is equivalent to a resistance to intertemporal substitution decreasing with age, $\Theta_1(c(s), s) = \rho_2(c(s), s)$.

Proof: The *common difference effect* is characterized by $\Theta_2(c(s), s) < 0$. Using the definition of Θ given by equation (3) and elementary calculations prove *a*). Similarly, the *absolute magnitude effect* is characterized by $\Theta_1(c(s), s) < 0$, by differentiating equation (2), we obtain *b*). *c*) is obtained from $\rho_2(c(s), s) < 0$. *d*) is obtained from *b*) and *c*), and

⁶Of course, on the other hand, the fact that these anomalies can be reproduced within a generalized model of time-consistent agents does not prove that agents are necessarily time consistent. I will return to this “time-inconsistent side” of the model in the next section

the application of Young’s theorem. \square

Proposition 3 gives a correspondence between the observable properties of the discount rate, which can be measured experimentally in the lab or on the field, and the properties of the utility function, that may be important for comparative statics analysis. This shows crucially that both *common difference* and *magnitude effect* depend on the sign and magnitude of some third derivative of the instantaneous utility function. More precisely, Proposition 3 i), ii) and iii) gives a condition that compares the log-derivative of a second-order derivative with the log-derivative of a first-order derivative. Because the log-derivative is a measure of the growth rate of a function, it means that, to observe *the common difference effect*, the growth rate according to s of the cross-derivative of w has to be higher than the growth rate according to s of the marginal felicity function of consumption. Identically, to observe *the absolute magnitude effect*, the growth rate according to c of the cross-derivative of w has to be higher than the growth rate according to c of the marginal felicity function of consumption.

What is striking is that these conditions are of the same mathematical nature as those given by Kimball (1990) to characterize the necessary conditions to observe precautionary savings. The only difference is that, in my model, the felicity function depends on two variables, and thus first-order and second-order derivatives are partial derivatives.

Proposition 2 d. may be the most remarkable part of the proposition. It stems from the internal consistency of the model. In particular, the instantaneous discount rate cannot be any function of c , t and s . It is the log-derivative according to s of the marginal “per-period” felicity of consumption, and the rate of absolute resistance to intertemporal substitution is also a log-derivative (according to c) of the same marginal “per-period” felicity of consumption).

If one believes in the absolute magnitude effect, in the context of the life-cycle model of consumption and savings, it implies that, everything else being equal, richer people will be more patient; then, one must necessarily believe that older people will be more resistant to intertemporal substitution. The problem is in direct relation with a well-known problem in finance, the problem of the evolution of risk aversion with age, since, under expected utility, parameter ρ will also characterize per period risk aversion.

In finance, it is common practice to consider that risk aversion increases with age. However, two famous theoretical papers, Samuelson (1969) and Merton (1969), have demonstrated that, under the standard expected utility dynamic model of portfolio choice with CRRA per-period utility function, the agent will have a risk aversion on wealth that is invariant with age. Reconciling the common practice with theory has since been an important research agenda (see Spaenjers and Spira (2015) for a recent survey and empirical results on the problem). Many explanations can be put forth, from assuming the convexity of absolute risk tolerance with regard to wealth (Gollier and Zeckhauser, 2002), to assuming loss aversion (Berkelaar et al., 2004). Unfortunately, the direct explanation that “per-period” absolute risk aversion can simply increase with age has not received much attention until now, even if there is some empirical evidence (Harrison et al., 2007, for example)⁷. This is probably because conventional wisdom has long wrongly associated non-stationarity with time inconsistency. Thus, in future research, the general model presented here can be a good starting point to study portfolio choices throughout the life

⁷In a setting focused on utility of wealth (and thus indirect utility of the one used in this paper), Nachman (1975) considered the possibility of temporally changing tastes and introduced a notion of *temporal risk aversion* on wealth.

cycle.

Regardless, it appears that assuming an absolute rate of aversion to intertemporal substitution that decreases with age is far from obvious from a life cycle perspective. The reverse assumption actually seems more natural. However, if it is the case, according to proposition 2 d., a negative *absolute magnitude effect* must also be discarded and replaced by a positive one.

Turning now to the interpretation of the common difference effect given at the beginning of the section, one may also wonder whether it accounts for empirical regularity in life-cycle consumption and savings. The work of Thurow (1969) has established the stylized fact that the life-cycle consumption profile is hump-shaped. Following Thurow himself, many works have attributed this property to the existence of a borrowing constraint and of buffer-stock savings in presence of earnings uncertainty (Carroll, 1997; Gourinchas and Parker, 2002, among others). But a more general and appealing explanation exists: life-time uncertainty. Following Yaari (1965), it is well known that a primitive for the subjective discount rate is the hazard rate of the mortality process, which is typically increasing and convex with age. Bommier (2013) and Drouhin (2015) have shown that various models of life-cycle consumption and savings with uncertain lifetime can explain the hump in the life-cycle consumption profile.

Regardless, the properties of the slope of a time-consistent agent's life-cycle consumption profile can be characterized.

Proposition 4. *The slope of a time-consistent agent's optimal consumption profile fulfils:*

$$\dot{c}(s) = \frac{r - \Theta(c_t(s), s)}{\rho(c_t(s), s)} \quad (16)$$

and

$$\ddot{c}(s) = \frac{-\Theta_2(c(s), s) - 2\dot{c}(s)\rho_2(c(s), s) - \dot{c}(s)^2\rho_1(c(s), s)}{\rho(c(s), s)} \quad (17)$$

Proof: Equation (16) stems directly from equations (5) and proposition 2. Equation (17) is just the derivative of (16) according to s . \square

Considering equations (17) and (16), it is obvious that the only way to obtain a hump-shaped consumption profile, without assuming a borrowing constraint, is to have a subjective discount rate that is increasing with age. Thus, the consumption profile will be increasing for young agents, reaching a maximum at middle age, and then decreasing. More specifically, equation (17) shows that the concavity of the consumption profile in the vicinity of the maximum consumption is only possible if $\Theta_2(c(s), s) > 0$, because in this vicinity the two other terms of the numerator are approximately zero ($\dot{c}(s) \approx 0$). However, when agents are young or old, the analysis of the concavity of c is richer because, in both cases, the slope of the consumption profile interacts with the first-order derivative of $\rho(c, s)$ to determine the concavity of c . Assuming that the per-period felicity function is DARA ($\rho_1(c(s), s) < 0$), then the third term of the denominator will be positive, implying an effect that will attenuate the concavity of c for both young and old people. Finally, assuming that the absolute coefficient of time resistance increases with age ($\rho_2(c(s), s) > 0$) (risk aversion increases with age), then the second term of the denominator will be negative for young people (reinforcing the concavity of c) and positive for old people (attenuating the concavity of c). The demonstration of this last effect is an innovation of this article, because it will only appear in a model that does

not multiplicatively separate the discount factor from a time-invariant utility function. Of course, assuming the opposite sign for the derivative of ρ , the sign of the effects will be reversed.

To conclude, it appears that the sign of the *absolute magnitude effect* and the interpretation given in this section of the *common difference effect* described by the experimental decision literature seem to contradict common stylized facts in the empirical literature on life-cycle consumption and savings. However, the theoretical model of this article, by assuming the possibility of a non-stationary intertemporal utility functional, provides a synthetical framework in which these two literatures can be connected. As such, it offers rigorous theoretical foundations for systematic empirical investigation of the effect of aging on fundamental concepts of economic theory, such as time preference and risk aversion.

5 A Naive, Time-Inconsistent Consumer's Effective Consumption Profile

With the intertemporal utility functional (1), the agent may be time inconsistent. When it is the case, the consumption path planned at date t will not be observable, because the agent will change her mind in the future. The properties of the planned consumption profile, given by proposition 1, are theoretically easy to compute. However, with the exception of the starting point $c_t(t)$, these properties may seem empirically useless because they will never be directly observed in the real world. Nevertheless, the intertemporal utility functional (1) includes a description of how the agent changes her mind. Thus, a theory of the consumption effectively consumed at each future date can be abstracted from the model.

In this section, I assume that the agent is time inconsistent, naive and characterized by a present bias. I will exploit the continuous time structure of the model to fully characterize and analyze the “effective consumption profile” of the agent, the one that is observable. I will follow an agent throughout her life cycle from the starting date⁸ t_0 , until her death at date T . I will suppose that at each instant $t \in [t_0, T]$, the agent is continuously planning her consumption path again for her remaining life interval $[t, T]$ (i.e. the agent solves the program \mathcal{P}_t) and effectively consumes $c_t(t)$, the initial value of each consumption profile planned at date t (i.e. the solution of each program \mathcal{P}_t).

Definition 5. *The effective consumption profile is defined as the function $\bar{c} : [t_0, T] \rightarrow \mathbb{R}^+$ with:*

$$\bar{c}(t) \stackrel{\text{def}}{=} c_t(t) \tag{18}$$

What are the properties of the effective consumption profile? To answer this question, it is first necessary to take a look at the differential constraint.

Proposition 5. *The effective capital profile fulfils the dynamic accumulation constraint.*

$$\forall t \in [t_0, T], \quad \dot{K}(t) = rK(t) - \bar{c}(t) \tag{19}$$

⁸Even if the birth date is normalized to 0, taking t_0 as a starting date is preferable for two reasons. First, agents effectively start deciding on their own consumption after their birth date. Second, for a clean exposition of problems relating to time consistency, it is always preferable to take dates as parameters.

and thus the life-cycle budgetary constraint.

$$\forall t \in [t_0, T], \quad \int_t^T \exp[-r(s-t)] \bar{c}(s) ds = K(t) \quad (20)$$

Proof: The differential budgetary constraint (19) is just an accounting equation. Taking equation (19) in s , and multiplying both sides by $\exp[-r(s-t)]$ and, then, integrating the obtained identity by parts on the interval $[t, T]$ gives (20). \square

As can be seen, the dynamic of the *effective capital profile* is very standard and fulfils all the usual properties of intertemporal choice models. It will allow to explicit the analytical properties of the *effective consumption profile*.

Proposition 6. *The effective consumption profile is characterized by :*

$$\bar{c}(t) = \frac{K(t)}{\int_t^T \exp \left[\int_t^s (G(t, \tau) - r) d\tau \right] ds} \quad (21)$$

and

$$\frac{\dot{\bar{c}}(t)}{\bar{c}(t)} = G(t, t) - \frac{\int_t^T \int_t^s \frac{\partial G(t, \tau)}{\partial t} d\tau \exp \left[\int_t^s (G(t, \tau) - r) d\tau \right] ds}{\int_t^T \exp \left[\int_t^s (G(t, \tau) - r) d\tau \right] ds} \quad (22)$$

Proof: Substituting (21) into the life-cycle budgetary constraint (6), we get:

$$K(t) = c_t(t) \int_t^T \exp \left[\int_t^s (G(t, \tau) - r) d\tau \right] ds \quad (23)$$

Using Definition 5, equation (21) is straightforward.

We denote,

$$I(t) = \int_t^T \exp \left[\int_t^s (G(t, \tau) - r) d\tau \right] ds \quad \text{with} \quad D_I = \{t; t_0 \leq t \leq T\} \quad (24)$$

Then equation (21) can be rewritten:

$$\bar{c}(t) = \frac{K(t)}{I(t)} \quad (25)$$

Taking the log-derivative of this equation, we get:

$$\frac{\dot{\bar{c}}(t)}{\bar{c}(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{I}(t)}{I(t)}$$

Since for all $t \in [t_0, T]$, the effective consumption path fulfils the differential constraint (19), we have:

$$\frac{\dot{K}(t)}{K(t)} = r - \frac{\bar{c}(t)}{K(t)} = r - \frac{1}{I(t)} \quad (26)$$

Using the Leibnitz formula on $I(t)$ twice, we get:

$$\dot{I}(t) = -1 - (G(t, t) - r)I(t) + \int_t^T \int_t^s \frac{\partial G(t, \tau)}{\partial t} d\tau \exp \left[\int_t^s (G(t, \tau) - r) d\tau \right] ds$$

Combining these three last equations, we get equation (22). \square

Equation (21) gives the value of the effective consumption at any date t . It is directly derived from the life-cycle budgetary constraints that fulfil the “planned consumption profile”. Equation (22) gives the dynamic of the effective consumption profile. It is particularly interesting because it allows to directly compare the slope of the planned consumption profile at its starting date, t , and the slope of the effective consumption profile at the same date, t . The sign of the difference between these two slopes is given by the sign of the integral term in equation ((22)). This term, which seems complicated at first, can be interpreted as a weighted average of the term $\int_t^s \frac{\partial G(t,\tau)}{\partial t} d\tau$ for each date s in $[t, T]$, the weight being the growth factor of the planned consumption profile, (at planning date t), for each date s in $[t, T]$ (i.e. $c_t(s) \exp[-r(s-t)]/c_t(t)$). Thus, when the growth rate of the planned consumption profile at any future date s is monotonic in the planning date t , because of the positivity of the exponential, the sign of the integral term will be determined, and thus the sign of the difference between the two slopes. We will see that this is especially true for the continuous time version of the quasi-hyperbolic discounting model given by Barro (1999). But first, the last general comparative properties of the effective and planned consumption profiles must be specified.

Proposition 7. *For a time-inconsistent agent, the planned consumption profile (at any date t) crosses the effective consumption profile at least twice. Moreover, if the agent is present biased, the planned consumption profile will always start by going below the effective one (i. e. $\dot{\bar{c}}(t) > \dot{c}_t(t)$).*

Proof: Both planned and effective consumption at date t satisfy the life-cycle budgetary constraints (respectively given by equations (6) and (20)). We thus have:

$$\forall t \in [t_0, T], \quad \int_t^T \exp[-r(s-t)](\bar{c}(s) - c_t(s))ds = 0 \quad (27)$$

By definition of the effective consumption profile, we necessarily have $c_t(t) = \bar{c}(t)$. If the agent is time inconsistent, there exists a date $s_0 \in (t, T]$ when $\bar{c}(s_0) - c_t(s_0) < 0$ or $\bar{c}(s_0) - c_t(s_0) > 0$. Then, because of equation (27), there necessarily exists a date $s_1 \in (t, T]$ when, respectively, $\bar{c}(s_1) - c_t(s_1) > 0$ or $\bar{c}(s_1) - c_t(s_1) < 0$. Since the per-period felicity function is strictly concave in c , c_t is necessarily continuous. Equation (21) shows that it will also be the case for the effective consumption profile. Thus, by continuity, we necessarily have a second date $\bar{s} \in (t, T]$ when $\bar{c}(\bar{s}) = c_t(\bar{s})$.

The last part of the proposition stems directly from Equation(22) and the Definition of present bias (Definition 4 in Section 3). \square

A parametric example can be taken to illustrate the results of Propositions 6 and 7. I will use the continuous time version of the quasi-hyperbolic discounting model given by Barro (1999):

$$V(c, s) = \int_t^T u(c(s)) \exp[-\alpha(s-t) - \phi(s-t)]ds \quad (28)$$

with $\phi(s-t) = -\beta \exp[-q(s-t)] + \beta$.

In this special case of the model, the per-period felicity function multiplicatively separates the discount factor and the per-period utility function, which is time invariant. Using the formalism of corollary 2, this parametric intertemporal utility functional belongs to the set B , but not to the set $TC(B)$, because the per-period felicity function is not multiplicatively separable in s and t . Following Definition 1, the instantaneous subjective

discount rate can be written $\Theta(c, s, t) = \alpha + q\beta \exp[-q(s - t)]$. Thus, the discount rate decreases with s , from $\alpha + q\beta$ when $s = t$ to α when s goes to infinity. The parameters q characterize the speed of convergence of the instantaneous subjective discount rate to its long term value α . If q is sufficiently high, the convergence is very rapid, approximating the discrete time quasi-hyperbolic discounting proposed by Laibson (1997).

For the purpose of simplicity and computability, I will assume that the per-period utility function is CRRA ($\rho(c, s, t)c_t(s) = \gamma = cst$).

With these assumptions, the growth rate of the planned consumption can be expressed:

$$\forall t \in [t_0, T], \forall s \in [t, T], \quad G(t, s) = \frac{\dot{c}_t(s)}{c_t(s)} = \frac{r - \alpha - \phi'(s - t)}{\gamma} \quad (29)$$

Contrary to the general model, the right-hand of the expression no longer depends on the level of the consumption. This implies that the planned consumption function is the solution of a linear homogenous differential equation with variable coefficient, and thus is explicitly solvable. By a simple calculation, we also obtain:

$$\forall t \in [t_0, T], \forall s \in [t, T], \quad \frac{\partial G(t, s)}{\partial t} = \frac{\phi''(s - t)}{\gamma} < 0 \quad (30)$$

This result provides the sign of the last term of the right-hand side of equation (22), and proves, according to Proposition 6 that, in this case, the effective consumption profile will always have a starting slope that is higher than that of the planned consumption profile.

The method to compute the effective consumption profile is to numerically solve integral $I(t)$ from equation (24) for all t in (t_0, T) and differential equation (26), which is, in this parametrical example, homogenous and linear with variable coefficient, and to compute $K(t)$ for all t in (t_0, T) . Then, exploiting equation (25), computing $\bar{c}(t)$ for all t in (t_0, T) is straightforward.

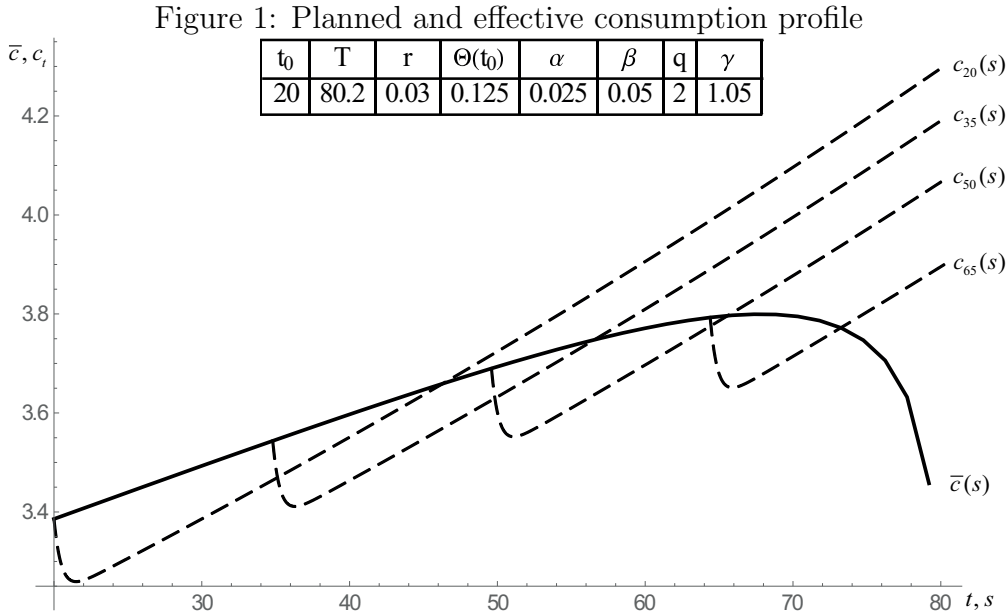


Figure 1 represents the effective consumption profile throughout the life cycle (with $t_0 = 20$ and $T = 80$) and the planned consumption profile at age 20, 35, 50, and 65, for

special numerical values. After a “short-term effect”, the planned consumption converges rapidly to a standard exponential discounting model with a CRRA utility function, characterized by a constant growth rate of the planned consumption. I choose parameters $q = 2$ to have a rapid convergence to the “long term regime”. I choose γ , the coefficient of relative resistance to intertemporal substitution to be “close” to one ($\gamma = 1.05$), so that the growth rate of the planned consumption can be approximated by the difference between the interest rate (r) and the instantaneous subjective discount rate (Θ). I take a value of 3% for the interest rate. I choose a value of 2.5% for α , the “long-term” instantaneous discount rate, such that the planned consumption will be asymptotically increasing. Finally, I take a value of 5% for parameters β . Thus, the instantaneous subjective discount rate at the planning date will be 12.5% ($\Theta(t, t) = 12.5\%$), implying that the planned consumption profile will be significantly decreasing at the planning date t and immediately after.

When considering Figure 1, it is striking to note that the planned and effective consumption profiles have such completely different shapes that it seems impossible to intuitively abstract the properties of the latter when knowing the properties of the former. Hopefully, Propositions 5, 6, and 7 allow to understand what is happening. Typically, when studying intertemporal choice models, one tends to focus on the slope of the consumption profile, which is given by the first-order condition of the maximization program. Of course, this condition is very important, but to fully characterize the consumption profile, the initial value of consumption at date t must be considered. This consumption $c_t(t)$ is chosen by the agent. However, this choice is constrained by the initial level of capital. If the agent chooses “too high” a level for $c_t(t)$, the planned consumption will not be sustainable because the capital will go to zero before the terminal date T . On the contrary, with “too low” a level for $c_t(t)$, the terminal value of the capital ($K(T)$) will be strictly positive, implying that some available resources are lost for consumption. Thus, the optimal value for $c_t(t)$ is precisely the one that will allow to fulfil the transversality condition, that is, in the absence of a bequest motive, the one that drives the capital to zero exactly at the terminal date T . This is precisely what is implied by the life-cycle budgetary constraint (6).

What happens for an agent that is not time consistent, in a way characterized by Barro (1999)’s utility functional? At date t , the agent plans her consumption throughout her life-cycle. She determines the optimal slope of consumption for all ages in $[t, T]$ and the initial level of consumption $c_t(t)$. After a “small” time interval ϵ , the agent realizes that she has changed her mind. She now wants a new consumption profile $c_{t+\epsilon}$, with a lower slope for all dates in $[t + \epsilon, T]$. However, she realizes that if she starts this new consumption profile, with the previously planned consumption for date $t + \epsilon$, she will end up at date T with a strictly positive level of capital. Thus, to fulfil the transversality condition, the agent has to start from a level of consumption at date $t + \epsilon$ that is necessarily higher than the one formerly planned (i.e. $c_{t+\epsilon}(t + \epsilon) > c_t(t + \epsilon)$). This reasoning is precisely the one that is behind proposition 6 and equation (22).

Figure 1 also illustrates the fact that the difference between the slope of the planned consumption profile and the effective one is subject to a horizon effect in the case of Barro (1999)’s quasi-hyperbolic discounting model. This is due to the integral term in equation (22) tending to zero, when t tends to T . Thus, when the agent gets older, the slope of the effective consumption profile gets closer to the initial slope of the planned consumption $G(t, t)$. That is why the effective consumption profile is hump-shaped.

Finally, Figure 1 illustrates Proposition 7. The planned consumption profile at any

date t starts from the effective consumption profile, goes immediately below it, but always finishes above it.

The example is illustrative, but the method applies to any possible functional form for the intertemporal utility functional. As seen, the life-cycle budgetary constraint plays a central role in all propositions in this section, proving that “the choice-based” methodology adopted is a powerful and necessary tool to make testable predictions for the consumption behavior of non-consistent agents.

6 Conclusion

By using the most general possible additive utility representation of intertemporal preferences, encompassing the possibility that the per-period felicity function varies with age and decision date, I have established three important and original sets of results:

1. I have shown, contrary to what is conventionally assumed, that exponential discounting is not the only additive intertemporal utility functional that is compatible with time consistency. In practice, I have demonstrated that any intertemporal utility functional of the form $V(c, t) = \int_t^T w(c(s), s)ds$ implies a time-consistent consumption behavior.
2. This implies that the possibility that the discount rate can vary with age or with the level of consumption cannot be rejected as implying irrational behavior (time-inconsistent behavior). This possibility should then be seriously investigated empirically. The results of this article provide a rigorous theoretical framework for doing so. In particular, the fact that both the instantaneous subjective rate of discount and coefficient of absolute resistance to intertemporal substitution (risk aversion) are log-derivatives of the marginal utility of consumption should be taken into account when exploring the effect of aging on intertemporal consumption behavior.
3. Finally, when using the model to describe a naive, time-inconsistent agent’s observable behavior, the effective consumption profile and its general mathematical properties can be characterized.

In line with Halevy (2015), but with a different methodology, I hope to have contributed to the clarification of the distinction between time consistency and stationarity. Halevy has done it, upstream, through rigorous axiomatic work, sustained by experimental evidence. In this article, I have done it, downstream, starting from a very general class of additive intertemporal utility functionals and solving dynamic optimization programs. Of course, the two approaches are complementary and should be investigated in parallel. An advantage of my “choice-based methodology” is that it is closest to the standard practice of applied economists. It allows to discuss functional forms (various forms of separability, sign of high-order derivatives, etc.), not only on the basis of axioms, but also taking into account empirical regularities outside the lab.

For too long, it has been considered that time consistency and stationarity were two sides of the same coin. In an aging economy, it is crucial to have a better understanding of how preferences evolve throughout the life cycle. For that purpose, stationarity must be discarded and alternative models explored. As such, this article is a first step in an applied research agenda, concerning not only consumption and savings, but all the related fields in which intertemporal choice plays a central role: labor, health, education, finance, and many others.

References

- Ainslie, G. (1975). Specious reward: A behavioural theory of impulsiveness and impulse control. *Psychological Bulletin* 82(4), 463–496.
- Allais, M. (1947). *Economie et Intérêt*. Imprimerie Nationale, Paris, France.
- Andersen, S., G. W. Harrison, M. I. Lau, and E. E. Rutström, E. E.m (2013). Discounting behaviour and the magnitude effect: Evidence from a field experiment in denmark. *Economica* 80(320), 670–697.
- Andersen, S., G. W. Harrison, M. I. Lau, and E. E. Rutström (2014). Discounting behavior: A reconsideration. *European Economic Review* 71, 15–33.
- Andreoni, J. and C. Sprenger (2012). Estimating time preferences from convex budgets. *The American Economic Review* 102(7), pp. 3333–3356.
- Barro, R. J. (1999). Ramsey meets Laibson in the neoclassical growth model. *Quarterly Journal of Economics* 114(4), 1125–1152.
- Berkelaar, A. B., R. Kouwenberg, and T. Post (2004). Optimal portfolio choice under loss aversion. *Review of Economics and Statistics* 86(4), 973–987.
- Blackorby, C., D. Nissen, D. Primont, and R. R. Russell (1973). Consistent intertemporal decision making. *The Review of Economic Studies* 40(2), 239–248.
- Blackorby, C., D. Primont, and R. R. Russel (1978). *Duality, Separability, and Functional Structure: Theory and Applications*. North-Holland.
- Bommier, A. (2013). Life-cycle preferences revisited. *Journal of the European Economic Association* 11(6), 1290–1319.
- Burness, H. S. (1976). A note on consistent naive intertemporal decision making and an application to the case of uncertain lifetime. *The Review of Economic Studies* 43(3), 547–549.
- Carroll, C. D. (1997). Buffer-stock saving and the life cycle/permanent income hypothesis. *The Quarterly Journal of Economics* 112(1), pp. 1–55.
- Drouhin, N. (2009). Hyperbolic discounting may be time consistent. *Economics Bulletin* 29(4), 2552–2558.
- Drouhin, N. (2015). A rank-dependent utility model of uncertain lifetime. *Journal of Economic Dynamics and Control* 53(0), 208 – 224.
- Epstein, L. G. and J. A. Hynes (1983). The rate of time preference and dynamic economic analysis. *Journal of Political Economy* 91(4), 611–635.
- Fishburn, P. C. and A. Rubinstein (1982). Time preference. *International Economic Review* 23(3), 677–694.
- Fisher, I. (1930). *The theory of interest*. Macmillan New York.

- Frederick, S., G. Loewenstein, and T. O'Donoghue (2002). Time discounting and time preference: A critical review. *Journal of Economic Literature* 40(2), 351–401.
- Gollier, C. (2001). *The economics of risk and time*. MIT press Cambridge.
- Gollier, C. and R. Zeckhauser (2005). Aggregation of heterogeneous time preferences. *Journal of Political Economy* 113(4), 878–896.
- Gollier, C. and R. J. Zeckhauser (2002, May). Horizon length and portfolio risk. *Journal of Risk and Uncertainty* 24(3), 195–212.
- Gourinchas, P.-O. and J. A. Parker (2002). Consumption over the life cycle. *Econometrica* 70(1), pp. 47–89.
- Halevy, Y. (2015). Time consistency: Stationarity and time invariance. *Econometrica* 83(1), 335–352.
- Hall, R. E. (1978). Stochastic implications of the life cycle-permanent income hypothesis: theory and evidence. *Journal of political economy* 86(6), 971–987.
- Harris, C. and D. Laibson (2001). Dynamic choices of hyperbolic consumers. *Econometrica* 69(4), 935–957.
- Harris, C. and D. Laibson (2013). Instantaneous gratification. *Quarterly Journal of Economics* 128(1), 205–248.
- Harrison, G. W., M. I. Lau, and E. E. Rutström (2007). Estimating risk attitudes in denmark: A field experiment. *Scandinavian Journal of Economics* 109(2), 341 – 368.
- Harrison, G. W., M. I. Lau, and E. Rutstrom (2005). Dynamic consistency in denmark: A longitudinal field experiment. *SSRN eLibrary*.
- Harvey, C. M. (1986). Value functions for infinite-period planning. *Management Science* 32(9), 1123–1139.
- Harvey, C. M. and L. P. Østerdal (2012). Discounting models for outcomes over continuous time. *Journal of Mathematical Economics* 48(5), 284 – 294.
- Kimball, M. S. (1990). Precautionary saving in the small and in the large. *Econometrica* 58(1), 53–73.
- Koopmans, T. C. (1960). Stationary ordinal utility and impatience. *Econometrica* 28(2), 287–309.
- Koopmans, T. C. (1972). Representation of preference orderings over time. In McGUIRE and RADNER (Eds.), *Decision and Organization, a Volume in Honor of Jacob Marschak.*, pp. 79–100.
- Laibson, D. (1994). *Hyperbolic discounting and consumption*. Ph. D. thesis, Massachusetts Institute of Technology.
- Laibson, D. (1997). Golden eggs and hyperbolic discounting. *Quarterly Journal of Economics* 112(2), 443–477.

- Laibson, D. (1998). Life-cycle consumptions and hyperbolic discount functions. *European Economic Review* 42, 861–871.
- Loewenstein, G. and D. Prelec (1992). Anomalies in intertemporal choice: Evidence and an interpretation. *The Quarterly Journal of Economics* 107(2), 573–597.
- Merton, R. C. (1969). Lifetime portfolio selection under uncertainty: The continuous-time case. *The review of Economics and Statistics*, 247–257.
- Nachman, D. C. (1975). Risk aversion, impatience, and optimal timing decisions. *Journal of Economic Theory* 11(2), 196–246.
- O’Donoghue, T. and M. Rabin (1999). Doing it now or later. *The American Economic Review* 89(1), 103–124.
- Pan, J., C. S. Webb, and H. Zank (2015). An extension of quasi-hyperbolic discounting to continuous time. *Games and Economic Behavior* 89(0), 43 – 55.
- Phelps, E. S. and R. A. Pollak (1968). On second-best national saving and game-equilibrium growth. *The Review of Economic Studies* 35(2), pp. 185–199.
- Prelec, D. (2004). Decreasing impatience: A criterion for non-stationary time preference and ‘hyperbolic’ discounting. *Scandinavian Journal of Economics* 106(3), 511 – 532.
- Ramsey, F. P. (1928). A mathematical theory of saving. *The Economic Journal* 38(152), pp. 543–559.
- Saint-Paul, G. (2011). *The tyranny of utility: Behavioral social science and the rise of paternalism*. Princeton University Press.
- Samuelson, P. A. (1937). A note on measurement of utility. *Review of Economic Studies* 4, 155–161.
- Samuelson, P. A. (1969). Lifetime portfolio selection by dynamic stochastic programming. *The review of economics and statistics*, 239–246.
- Sayman, S. and A. Onculer (2009). An investigation of time inconsistency. *Management Science* 55(3), 470–482.
- Spaenjers, C. and S. M. Spira (2015). Subjective life horizon and portfolio choice. *Journal of Economic Behavior & Organization* 116, 94 – 106.
- Strotz, R. H. (1956). Myopia and inconsistency in dynamic utility maximisation. *Review of Economic Studies* 23(2), 165–180.
- Thaler, R. (1981). Some empirical evidence on dynamic inconsistency. *Economics Letters* 8, 201–207.
- Thaler, R. H. and C. R. Sunstein (2003). Libertarian paternalism. *The American Economic Review* 93(2), 175–179.
- Thurow, L. C. (1969). The optimum lifetime distribution of consumption expenditures. *The American Economic Review* 59(3), pp. 324–330.

- Wakker, P. P. (1989). *Additive representations of preferences: A new foundation of decision analysis*, Volume 4. Kluwer Academic Publishers.
- Yaari, M. E. (1964). On the consumer's lifetime allocation process. *International Economic Review* 5(3), 304–317.
- Yaari, M. E. (1965). Uncertain lifetime, life insurance, and the theory of the consumer. *The Review of Economic Studies* 32(2), 137–150.